

## Poles of intégrale tritronquée and anharmonic oscillators. A WKB approach

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2010 J. Phys. A: Math. Theor. 43 095201

(<http://iopscience.iop.org/1751-8121/43/9/095201>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.158

The article was downloaded on 03/06/2010 at 08:57

Please note that [terms and conditions apply](#).

# Poles of intégrale tritronquée and anharmonic oscillators. A WKB approach\*

**Davide Masoero**

Mathematical Physics Sector, SISSA, Trieste, Italy

E-mail: [dmasoero@gmail.com](mailto:dmasoero@gmail.com)

Received 7 October 2009, in final form 7 December 2009

Published 9 February 2010

Online at [stacks.iop.org/JPhysA/43/095201](http://stacks.iop.org/JPhysA/43/095201)

## Abstract

Poles of solutions to the Painlevé-I equations are intimately related to the theory of the cubic anharmonic oscillator. In particular, poles of intégrale tritronquée are in bijection with cubic oscillators that admit the simultaneous solutions of two quantization conditions. We analyze this pair of quantization conditions by means of a suitable version of the complex WKB method.

PACS numbers: 02.30.Hq, 02.30.Zz, 03.65.Sq

(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

The aim of this paper is to study the distribution of poles of the solutions  $y = y(z)$  to the Painlevé-I equation (P-I)

$$y'' = 6y^2 - z, \quad z \in \mathbb{C},$$

with particular attention to the poles of the *intégrale tritronquée*.

As is well known, any local solution of P-I extends to a global meromorphic function  $y(z)$ ,  $z \in \mathbb{C}$ , with an essential singularity at infinity [GLS00]. Global solutions of P-I are called Painlevé-I transcendents, since they cannot be expressed via elementary functions or classical special functions [Inc56]. The *intégrale tritronquée* is a special P-I transcendent, which was discovered by Boutroux in his classical paper [Bou13] (see [JK88, Kit94] for a modern review). Boutroux characterized the *intégrale tritronquée* as the unique solution of P-I with the following asymptotic behavior at infinity:

$$y(z) \sim -\sqrt{\frac{z}{6}}, \quad \text{if } |\arg z| < \frac{4\pi}{5}.$$

Nowadays, the Painlevé-I equation is studied in many areas of mathematics and physics. Indeed, it is remarkable that special solutions of P-I describe semiclassical asymptotics

\* Ai miei zii Ivan e Silvia.

of a wealth of different problems (see [Kap04] and references therein). In particular, in [DGK09] it is discovered that the intégrale tritronquée provides the universal correction to the dispersionless limit of solutions to the focusing nonlinear Schrödinger equation.

Theoretical and numerical evidences led the authors of [DGK09] to the following inspiration.

**Conjecture.** *If  $a \in \mathbb{C}$  is a pole of the intégrale tritronquée then  $|\arg a| \geq \frac{4\pi}{5}$ .*

Following the *isomonodromic* approach to P-I [Kap04] (see also [FMZ92]), any solution  $y(z)$  gives rise to an *isomonodromic deformation* of the following linear equation with an irregular singularity:

$$\overrightarrow{\Phi}_\lambda(\lambda, z) = \begin{pmatrix} y'(z) & 2\lambda^2 + 2\lambda y(z) - z + 2y^2(z) \\ 2(\lambda - y(z)) & -y'(z) \end{pmatrix} \overrightarrow{\Phi}(\lambda, z).$$

The deformation of the equation is manifestly singular at every pole  $a \in \mathbb{C}$  of  $y$ ; however, in theorem 2 we show that at the singularity this equation can be replaced with a simpler one, which has the same *monodromy data* (cf [IN86] for Painlevé II). This is the following Schrödinger equation with cubic potential:

$$\frac{d^2\psi(\lambda)}{d\lambda^2} = V(\lambda; a, b)\psi(\lambda), \quad V(\lambda; a, b) = 4\lambda^3 - 2a\lambda - 28b.$$

Here  $a$  is the location of the pole of  $y$  and  $b$  is a complex number entering into the Laurent expansion of  $y$  around  $a$  (see formula (7) below).

The isomonodromy property implies that there exists a natural injective map  $\mathcal{M}$  from the space of solutions of P-I to the space of *monodromy data* of the above equations (see lemma 2), while the Schrödinger equation defines naturally a map  $\mathcal{T}$  from the space of cubic potentials to the space of monodromy data.

Our first main result is theorem 3, which states that  $a \in \mathbb{C}$  is a pole of  $y(z)$  *if and only if* there exists  $b \in \mathbb{C}$  such that  $\mathcal{M}(y) = \mathcal{T}(V(\lambda; a, b))$ .

In particular, due to the special monodromy data related to the intégrale tritronquée (see theorem 1, due to Kapaev), we will show that the poles of the intégrale tritronquée are in bijection with the simultaneous solutions of two different *quantization conditions*.

The above approach naturally embeds the study of poles of Painlevé-I transcendents into Nevanlinna's theory of branched coverings of the sphere and the complex WKB method.

The beautiful theory of Nevanlinna (see [Nev70, Elf34]) relates bijectively the Schrödinger equations to a polynomial potential to the branched coverings of the sphere with logarithmic branch points, considered up to conformal equivalence. Using this theory we are able to prove the surjectivity of the map  $\mathcal{M}$  (see theorem 5).

Moreover, Nevanlinna's theory provides the poles of any solution of P-I with an unexpected and remarkable rich structure. In particular, poles of the tritronquée solution can be labeled by the monodromy of coverings of the Riemann sphere with three logarithmic branch points. In a subsequent paper, we will use this topological description to complete the WKB analysis of the present paper.

A WKB analysis of P-I was developed in [KT05], but has never been applied to the direct study of the distributions of poles. To achieve such a goal we follow Fedoryuk's approach (see [Fed93]) to the complex WKB theory, and in the classification theorem we give a complete topological classification of the *Stokes complexes* of all cubic potentials. As a consequence of the classification theorem, we obtain our second main result: all polynomials whose monodromy data, in the WKB approximation, are the monodromy data of the intégrale

tritonquée have the same topological type of Stokes complex and satisfy a pair of *Bohr–Sommerfeld* quantization conditions, namely system (25). In particular, in this way we reproduce the conditions obtained by Boutroux, through a completely different approach, in his study of the asymptotic distributions of the poles of the intégrale tritonquée.

*A priori*, the WKB method is expected to give an approximation of poles  $z = a$  for  $a$  sufficiently large. Surprisingly our approach proves to be numerically very efficient also for poles close to the origin, see table 2 below.<sup>1</sup>

The paper is organized as follows. In section 2 we derive the Schrödinger equation associate with P-I and study thoroughly its relations with poles of P-I transcendents. Section 3 is devoted to the topological classification of Stokes complexes. In section 4 we calculate the monodromy data in the WKB approximation, we derive the correct Bohr–Sommerfeld conditions for the poles of tritonquée, and we introduce the ‘small parameter’ of the approximation. In section 5 we obtain an asymptotic description of poles of the *intégrale tritonquée*. In appendices A and B we prove some theorems regarding the WKB functions that are used in sections 2 and 3.

## 2. Poles and cubic oscillators

We review some well-known facts regarding the isomonodromic approach to the P-I equation and analyze the isomonodromic deformation in a neighborhood of the singularities.

### 2.1. P-I as an isomonodromic deformation

P-I is equivalent to the compatibility condition of the following system of linear ODEs:

$$\vec{\Phi}_\lambda(\lambda, z) = \begin{pmatrix} y'(z) & 2\lambda^2 + 2\lambda y(z) - z + 2y^2(z) \\ 2(\lambda - y(z)) & -y'(z) \end{pmatrix} \vec{\Phi}(\lambda, z) \quad (1)$$

$$\vec{\Phi}_z(\lambda, z) = - \begin{pmatrix} 0 & 2y(z) + \lambda \\ 1 & 0 \end{pmatrix} \vec{\Phi}(\lambda, z). \quad (2)$$

The precise meaning of the word *compatibility* is given by the following.

**Lemma 1.** Fix  $z_0, \lambda_0$  and the Cauchy data  $y(z_0), y'(z_0)$  and  $\vec{\Phi}(\lambda_0, z_0)$ . Let  $U_{z_0}$  be any simply connected neighborhood of  $z_0$ . Then  $y(z)$  satisfies the Painlevé first equation in  $U_{z_0}$  iff the system (1), (2) has a solution  $\forall(\lambda, z) \in \mathbb{C} \times U_{z_0}$ . Moreover, the solution is unique.

**Proof.** See [Kap04]. □

In this subsection we suppose that we have fixed a solution  $y$  of P-I and a simply connected region  $U$  such that  $y|_U$  is holomorphic.

We are now going to define the important concepts of *monodromy data* and *isomonodromic deformation* of equation (1). For this reason, we have to introduce some particular solutions of system (1), (2), to be uniquely defined by the asymptotic behavior for  $\lambda \rightarrow \infty$ .

Fix  $k \in \mathbb{Z}_5 = \{-2, \dots, 2\}$  and the branch of  $\lambda^{\frac{1}{5}}$  in such a way that  $\text{Re}\lambda^{\frac{5}{2}} \rightarrow +\infty$  as  $|\lambda| \rightarrow \infty, \arg \lambda = \frac{2\pi k}{5}$ . Then (see [Kap04]) for any  $y$  solution of P-I, there exists a unique

<sup>1</sup> In a recent paper [Nov09] Novokshenov studied numerically the distribution of poles of the intégrale tritonquée by means of the Padé approximation technique.

solution  $\vec{\Phi}_k(\lambda, z)$  of (1), (2) such that

$$\lim_{\substack{\lambda \rightarrow \infty \\ |\arg \lambda - 2\frac{\pi k}{5}| < \frac{3\pi}{5} - \varepsilon}} e^{+\frac{4}{5}\lambda^{\frac{5}{2}} - \frac{1}{2}z\lambda^{\frac{1}{2}}} \begin{pmatrix} \lambda^{-\frac{1}{4}} & 0 \\ 0 & \lambda^{+\frac{1}{4}} \end{pmatrix} \vec{\Phi}_k(\lambda) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \forall z \in U, \quad (3)$$

where  $\lambda^{\frac{1}{4}}$  is defined globally on the complex plane minus the negative real axis, and is positive on the positive real axis. Note that, depending on  $k \in \mathbb{Z}_5$ ,  $(\lambda^{\frac{1}{4}})^2$  may not be equal to  $\lambda^{\frac{1}{2}}$ . Here and in the following, if not otherwise stated,  $\varepsilon$  is an arbitrarily small positive number.

From the asymptotics (3) it follows that  $\vec{\Phi}_k(\lambda, z)$  and  $\vec{\Phi}_{k+1}(\lambda, z)$  are linearly independent for any  $k \in \mathbb{Z}_5$  and the following equality holds true:

$$\vec{\Phi}_{k-1}(\lambda) = \vec{\Phi}_{k+1}(\lambda) + \sigma_k(z) \vec{\Phi}_k(\lambda), \quad (4)$$

where  $\sigma_k(z)$  is an analytic function of  $z$ , for any  $k \in \mathbb{Z}_5$ .

**Definition 1.** Fixed  $z$ , we call  $\sigma_k$  the  $k$ th Stokes multiplier of equation (1) and the set of all five Stokes multipliers the monodromy data of (1). The problem of calculating the monodromy data is called the direct monodromy problem.

Stokes multipliers are very important for our analysis and we list their main properties in the following.

**Lemma 2.** Let  $\sigma_k(z), k \in \mathbb{Z}_5$  be defined as above. Then

- (i) equation (2) is an isomonodromic deformation of equation (1), i.e.  $\frac{d\sigma_k(z)}{dz} = 0$ .
- (ii) The numbers  $\sigma_k, k \in \mathbb{Z}_5$  satisfy the following system of algebraic equations:

$$1 + \sigma_k \sigma_{k+1} = -i \sigma_{k+3}, \quad k \in \mathbb{Z}_5. \quad (5)$$

**Proof.** See [Kap04]. □

Observe that only three of the algebraic equations (5) are independent.

**Definition 2.** We denote  $V$  the algebraic variety of quintuplets of complex numbers satisfying (5) and call admissible monodromy data the elements of  $V$ . Due to lemma 2, equations (4) define the following map:

$$\mathcal{M} : \{P\text{-I transcendentals}\} \rightarrow V.$$

**Lemma.**  $\mathcal{M}$  is injective.

**Proof.** See [Kap04]. □

We end the section with a result of Kapaev, which completely characterizes the intégrale tritronquée in terms of Stokes multipliers.

**Theorem 1** (Kapaev). The image under  $\mathcal{M}$  of the intégrale tritronquée is the monodromy data uniquely characterized by the following equalities:

$$\sigma_2 = \sigma_{-2} = 0. \quad (6)$$

**Proof.** See [Kap04]. □

2.2. Poles of  $y$ : cubic oscillator

So far we dealt with system (1), (2) in a region  $U$  which does not contain any pole of  $y(z)$ . Indeed, the situation at a pole is different, for equation (1) makes no sense.

However, we show that any solution  $\vec{\Phi}(\lambda, z)$  of system (1), (2) is meromorphic in all the  $z$ -plane; moreover, a pole  $z = a$  of  $y(z)$  is also a pole of  $\vec{\Phi}(\lambda, z)$  and the residue at the pole of its second component satisfies the scalar equation of Schrödinger type (8).

In order to be able to describe the local behavior of  $\vec{\Phi}(\lambda, z)$  near a pole  $a$  of  $y(z)$ , we have to know the local behavior of  $y(z)$  close to the same point  $a$ .

**Lemma 3** (Painlevé). *Let  $a \in \mathbb{C}$  be a pole of  $y$ . Then in a neighborhood of  $a$ ,  $y$  has the following convergent Laurent expansion:*

$$y(z) = \frac{1}{(z-a)^2} + \frac{a(z-a)^2}{10} + \frac{(z-a)^3}{6} + b(z-a)^4 + \sum_{j \geq 5} c_j(a, b)(z-a)^j \tag{7}$$

where  $b$  is some complex number and  $c_j(a, b)$  are real polynomials in  $a$  and  $b$ , not depending on the particular solution  $y$ .

Conversely, fixed arbitrary  $a, b \in \mathbb{C}$ , the above expansion has a non zero radius of convergence and solves P-I.

**Proof.** See [GLS00]. □

**Definition 3.** We define the map

$$\mathcal{L} : \mathbb{C}^2 \rightarrow \{P\text{-I transcendentals}\}.$$

$\mathcal{L}(a, b)$  is the unique analytic continuation of the Laurent expansion (7).

We have already collected all elements necessary to formulate the important

**Theorem 2.** Fix a solution  $y$  of P-I and let  $\Phi_k^{(i)}(\lambda, z)$ ,  $i = 1, 2$ ,  $k \in \mathbb{Z}_5$  be the  $i$ th component of  $\vec{\Phi}_k(\lambda, z)$ . Then

- (i)  $\vec{\Phi}_k(\lambda, z)$  is a meromorphic function of  $z$ . All the singularities are double poles. Moreover,  $a \in \mathbb{C}$  is a pole of  $\vec{\Phi}_k(\lambda, z)$  iff it is a pole of  $y$ .
- (ii) If  $a \in \mathbb{C}$  is a pole of  $y$  then

$$\Psi_k(\lambda) = \lim_{z \rightarrow a} (z-a)\Phi_k^{(2)}(\lambda, z)$$

is an entire function of  $\lambda$ . It satisfies the following Schrödinger equation with cubic potential:

$$\frac{d^2 \Psi_k(\lambda)}{d\lambda^2} = (4\lambda^3 - 2a\lambda - 28b)\Psi_k(\lambda), \tag{8}$$

where  $b \in \mathbb{C}$  is the coefficient entering into the Laurent expansion (7) of  $y$  around  $a$ .

- (iii) If  $\lambda^{\frac{1}{2}}$  and  $\lambda^{\frac{1}{4}}$  are chosen as in asymptotics (3), then  $\forall \varepsilon > 0$

$$\lim_{\lambda \rightarrow \infty, |\lambda - \frac{2\pi k}{5}| < \frac{3\pi}{5} - \varepsilon} \lambda^{\frac{3}{4}} e^{+\frac{4}{5}\lambda^{\frac{5}{2}} - \frac{1}{2}a\lambda^{\frac{1}{2}}} \Psi_k(\lambda) = i. \tag{9}$$

- (iv) Equation (8) possesses the same monodromy data as equation (1), i.e.

$$\Psi_{k-1}(\lambda) = \Psi_{k+1}(\lambda) + \sigma_k \Psi_k(\lambda).$$

**Proof.** (i) From the Laurent expansion (7), it is easily seen that a pole  $a$  of  $y$  is a fuchsian singularity with trivial monodromy of equation (2). In particular, the following Laurent expansions of  $\Phi_k^{(i)}(\lambda, z)$  are valid:

$$\begin{aligned} \Phi_k^{(2)}(\lambda, z) &= \frac{\psi_k(\lambda)}{(z-a)} \left( 1 - \frac{\lambda}{2}(z-a)^2 \right) + \varphi_k(\lambda)(z-a)^2 + O((z-a)^3), \\ \Phi_k^{(1)}(\lambda, z) &= \frac{\psi_k(\lambda)}{(z-a)^2} \left( 1 + \frac{\lambda}{2}(z-a)^2 \right) - 2\varphi_k(\lambda)(z-a) + O((z-a)^2). \end{aligned} \tag{10}$$

Expansions (10) show that  $\vec{\Phi}_k(\lambda, z)$  is meromorphic in a neighborhood of the point  $a$  and this point is a pole of order not greater than 2.

(ii), (iii) The proof is in appendix B.

(iv) Since the functions  $(z-a)\Phi_k^{(2)}$  satisfy equations (4) for any  $z$  with constant Stokes multipliers, then their limit, i.e. the functions  $\Psi_k(\lambda)$ , satisfy the same equations.  $\square$

**Definition 4.** We call any cubic polynomial of the form  $V(\lambda; a, b) = 4\lambda^3 - 2a\lambda - 28b$  a cubic potential. The above formula identifies the space of cubic potentials with  $\mathbb{C}^2 \ni (a, b)$ .

We define the map

$$\mathcal{T} : \mathbb{C}^2 \rightarrow V.$$

$\mathcal{T}(a, b)$  is the monodromy data of equation (8).

Theorem 2 has the following.

**Corollary.**  $\mathcal{M} \circ \mathcal{L} = \mathcal{T}$ .

The above corollary implies

**Theorem 3.** Let  $y$  be any solution of P-I. Then  $a \in \mathbb{C}$  is a pole of  $y$  iff there exists  $b \in \mathbb{C}$  such that  $\mathcal{M}(y) = \mathcal{T}(a, b)$ .

We finish this section with a theorem from Nevanlinna’s theory [Nev70], which implies the surjectivity of the map  $\mathcal{M}$ .

**Theorem 4.** The map  $\mathcal{T}$  is surjective. The preimage of any admissible monodromy data is a countable infinite subset of the space of cubic potentials.

**Proof.** See [Elf34].  $\square$

As a consequence of the above theorem we have the following.

**Theorem 5** (stated in [KK93]). The map  $\mathcal{M}$  is bijective: solutions of P-I are in 1-to-1 correspondence with admissible monodromy data.

Theorem 3 shows that the distribution of poles of P-I transcendents is a part of the theory of *anharmonic oscillators*, which has been the object of intense study since the seminal papers [BW68, Sim70].

**Remark.** In the theory of anharmonic oscillators a special importance is given to the vanishing of some Stokes multipliers. For a given  $k \in \mathbb{Z}_5$ , the problem is to find all  $(a, b) \in \mathbb{C}^2$  such that the Stokes multiplier  $\sigma_k$  of equation (8) vanishes. This is called the  $k$ th lateral connection problem. Since fixed  $a$ , there exists a discrete number of solutions to any lateral connection problem, equation  $\sigma_k = 0$  is referred to as a *quantization condition*.

As a consequence of theorems 1 and 3, we have the following.

**Corollary.** *The point  $a \in \mathbb{C}$  is a pole of the intégrale tritronquée if and only if there exists  $b \in \mathbb{C}$  such that the Schrödinger equation with the cubic potential  $V(\lambda; a, b)$  admits the simultaneous solution of two different quantization conditions, namely  $\sigma_{\pm 2} = 0$ .*

### 2.3. Asymptotic values

As was previously observed, Stokes multipliers are defined by particular normalized solutions of equations (1) and (8). Following Nevanlinna, we define the monodromy data of equation (8) in a more invariant way.

**Definition 5.** *Let  $\{\varphi, \chi\}$  be a basis of solution of (8).*

*We call*

$$w_k(\varphi, \chi) = \lim_{\substack{\lambda \rightarrow \infty \\ |\arg \lambda - \frac{2\pi k}{5}| < \frac{\pi}{5} - \varepsilon}} \frac{\varphi(\lambda)}{\chi(\lambda)} \in \mathbb{C} \cup \infty, \quad k \in \mathbb{Z}_5. \tag{11}$$

*the  $k$ th asymptotic value.*

We collect the main properties of the asymptotic values in the following.

**Lemma 4.**

(i) *Let  $\varphi' = a\varphi + b\chi$  and  $\chi' = c\varphi + d\chi$ ,  $a, b, c, d \in \mathbb{C}$ . Then*

$$w_k(\varphi', \chi') = \frac{aw_k(\varphi, \chi) + b}{cw_k(\varphi, \chi) + d}. \tag{12}$$

(ii)  *$w_{k-1}(\varphi, \chi) = w_{k+1}(\varphi, \chi)$  iff  $\sigma_k = 0$ .*

(iii)  *$w_{k+1}(\varphi, \chi) \neq w_k(\varphi, \chi)$*

**Proof.** See [Elf34]. □

Making use of equation (11), given the Stokes multipliers it is possible to calculate the asymptotic values. The converse is also true. In particular, the asymptotic values associated to the tritronquée intégrale can be chosen to be

$$w_0 = 0, \quad w_1 = w_{-2} = 1, \quad w_2 = w_{-1} = \infty. \tag{13}$$

### 3. Stokes complexes

In the complex WKB method a prominent role is played by the *Stokes and anti-Stokes lines*, and in particular by the topology of the *Stokes complex*, which is the union of the Stokes lines.

The main result of this section is the classification theorem, where we show that the topological classification of Stokes complexes divides the space of cubic potentials into seven disjoint subsets.

To avoid any misunderstanding, we stress that by our convention Stokes lines are level curves of the real part of the action.

**Remark.** To simplify the notation and avoid repetitions, we study the Stokes lines only. Every single statement in the following section remains true if the word Stokes is replaced with the word anti-Stokes, provided in equation (14) the angles  $\varphi_k$  are replaced with the angles  $\varphi_k + \frac{\pi}{5}$ .



**Definition 6.** A simple (resp. double, resp. triple) zero  $\lambda_i$  of  $V(\lambda) = V(\lambda; a, b)$  is called a simple (resp. double, resp. triple) turning point. All other points are called generic.

Fix a generic point  $\lambda_0$  and a choice of the sign of  $\sqrt{V(\lambda_0)}$ . We call action the analytic function

$$S(\lambda_0, \lambda) = \int_{\lambda_0}^{\lambda} \sqrt{V(u)} \, du$$

defined on the universal covering of  $\lambda$ -plane minus the turning points.

Let  $\tilde{i}_{\lambda_0}$  be the level curve of the real part of the action passing through a lift of  $\lambda_0$ . Call its projection to the punctured plane  $i_{\lambda_0}$ . Since  $i_{\lambda_0}$  is a one dimensional manifold, it is diffeomorphic to a circle or to a line. If  $i_{\lambda_0}$  is diffeomorphic to the real line, we choose one diffeomorphism  $i_{\lambda_0}(x)$ ,  $x \in \mathbb{R}$  in such a way that the continuation along the curve of the imaginary part of the action is a monotone increasing function of  $x \in \mathbb{R}$ .

**Lemma 5.** Let  $\lambda_0$  be a generic point. Then  $i_{\lambda_0}$  is diffeomorphic to the real line, the limit  $\lim_{x \rightarrow +\infty} i_{\lambda_0}(x)$  exists (as a point in  $\mathbb{C} \cup \infty$ ) and it satisfies the following dichotomy:

- (i) Either  $\lim_{x \rightarrow +\infty} i_{\lambda_0}(x) = \infty$  and the curve is asymptotic to one of the following rays of the complex plane:

$$\lambda = \rho e^{i\varphi_k}, \quad \varphi_k = \frac{(2k+1)\pi}{5}, \quad \rho \in \mathbb{R}^+, \quad k \in \mathbb{Z}_5, \quad (14)$$

- (ii) or  $\lim_{x \rightarrow +\infty} i_{\lambda_0}(x) = \lambda_i$ , where  $\lambda_i$  is a turning point.

Furthermore,

- (i) if  $\lim_{x \rightarrow \pm\infty} i_{\lambda_0}(x) = \infty$ , then the asymptotic ray in the positive direction is different from the asymptotic ray in the negative direction.
- (ii) Let  $\varphi_k, k \in \mathbb{Z}_5$  be defined as in equation (14). Then  $\forall \varepsilon > 0, \exists K \in \mathbb{R}^+$  such that if  $\varphi_{k-1} + \varepsilon < \arg \lambda_0 < \varphi_k - \varepsilon$  and  $|\lambda_0| > K$ , then  $\lim_{x \rightarrow \pm\infty} i_{\lambda_0}(x) = \infty$ . Moreover the asymptotic rays of  $i_{\lambda_0}$  are the ones with arguments  $\varphi_k$  and  $\varphi_{k-1}$ .

**Proof.** See [Str84]. □

**Definition 7.** We call Stokes line the trajectory of any curve  $i_{\lambda_0}$  such that there exists at least one turning point belonging to its boundary.

We call a Stokes line internal if  $\infty$  does not belong to its boundary.

We call Stokes complex the union of all the Stokes lines together with the turning points.

We state all important properties of the Stokes lines in the following.

**Theorem 6.** The following statements hold true:

- (i) The Stokes complex is simply connected. In particular, the boundary of any internal Stokes line is the union of two different turning points.
- (ii) Any simple (resp. double, resp. triple) turning point belongs to the boundary of three (resp. 4, resp 5) Stokes lines.
- (iii) If a turning point belongs to the boundary of two different non-internal Stokes lines then these lines have different asymptotic rays.
- (iv) For any ray with the argument  $\varphi_k$  as in equation (14), there exists a Stokes line asymptotic to it.

**Proof.** See [Str84]. □

3.1. Topology of stokes complexes

In what follows, we give a complete classification of the Stokes complexes, with respect to the orientation preserving homeomorphisms of the plane.

We define the map  $L$  from the  $\lambda$ -plane to the interior of the unit disk as

$$L: \mathbb{C} \rightarrow D_1 \quad L(\rho e^{i\varphi}) = \frac{2}{\pi} e^{i\varphi} \arctan \rho. \tag{15}$$

The image under the map  $L$  of the Stokes complex is naturally a decorated graph embedded in the closed unit disk. The vertices are the images of the turning points and the five points on the boundary of the unit disk with arguments  $\varphi_k$ , with  $\varphi_k$  as in equation (14). The bonds are obviously the images of the Stokes lines. We call the first set of vertices *internal* and the second set of vertices *external*. External vertices are decorated with the numbers  $k \in \mathbb{Z}_5$ . We denote  $S$  the decorated embedded graph just described. Note that due to theorem 6 (iii), there exists not more than one bond connecting two vertices.

The combinatorial properties of  $S$  are described in the following.

**Lemma 6.** *S possesses the following properties:*

- (i) *the sub-graph spanned by the internal vertices has no cycles.*
- (ii) *Any simple (resp. double, resp. triple) turning point has valency 3 (resp. 4, resp. 5).*
- (iii) *The valency of any external vertex is at least 1.*

**Proof.**

- (i) Theorem 6 part (i)
- (ii) Theorem 6 part (ii)
- (iii) Theorem 6 part (iv). □

**Definition 8.** *We call an admissible graph any decorated simple graph embedded in the closure of the unit disk with three internal vertices and five decorated external vertices, such that (i) the cyclic order inherited from the decoration coincides with the one inherited from the counter-clockwise orientation of the boundary, and (ii) it satisfies all the properties of lemma 6. We call two admissible graphs equivalent if there exists an orientation-preserving homeomorphism of the disk mapping one graph into the other.*

**Theorem 7. Classification theorem**

*All equivalence classes of admissible maps are, modulo a shift  $k \rightarrow k + m, m \in \mathbb{Z}_5$  of the decoration, the ones depicted in figure 1.*

**Proof.** Let us start analyzing the admissible graphs with three internal vertices and no internal edges.

Any internal vertex is adjacent to a triplet of external vertices. Due to the Jordan curve theorem, there exists an internal vertex, say  $\lambda_0$ , adjacent to a triplet of non consecutive external vertices. Performing a shift, they can be chosen to be the ones labeled by 0, 2, -1. Call the respective edges  $e_0, e_{-1}, e_2$ .

The disk is cut into three disjoint domains by those three edges. No internal vertices can belong to the domain cut by  $e_0$  and  $e_4$ , since it could be adjacent only to two external vertices, namely the ones labeled with 0 and -1. By similar reasoning it is easy to show that one and only one vertex belongs to each remaining domains.

Such an embedded graph is equivalent to the graph (3 0 0).

Classifications for all other cases may proved by similar methods. □

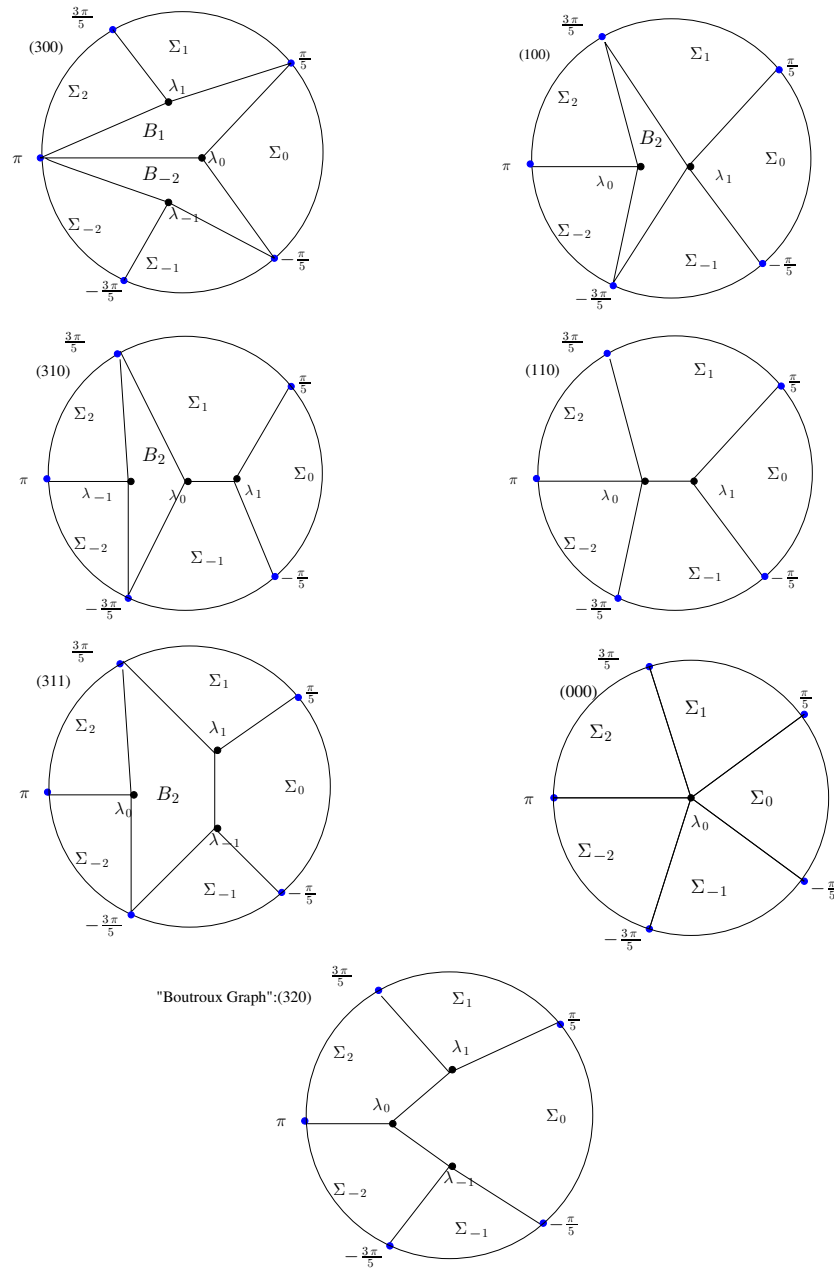


Figure 1. All the equivalence classes of admissible graphs.

The equivalence classes are encoded by a triplet of numbers  $(a b c)$ :  $a$  is the number of simple turning points,  $b$  is the number of internal Stokes lines, while  $c$  is a progressive number, distinguishing non-equivalent graphs with same  $a$  and  $b$ . Some additional information shown in figure 1 will be explained in the next section.

**Remark.** For any admissible graph there exists a real polynomial with an equivalent Stokes complex.

**Remark.** Note that the automorphism group of every graph in figure 1 is trivial. Therefore the unlabeled vertices can be labeled. In the following we will label the turning points as in figure 1. We denote “Boutroux graph” the graph (3 2 0).

### 3.2. Stokes sectors

In the  $\lambda$ -plane the complement of the Stokes complex is the disjoint union of a finite number of connected and simply-connected domains, each of them called a *sector*.

Combining theorem 6 and the classification theorem we obtain the following.

**Lemma 7.** *All the curves  $i_{\lambda_0}$ , with  $\lambda_0$  belonging to a given sector, have the same two asymptotic rays. Moreover, two different sectors have different pairs of asymptotic rays.*

*For any  $k \in \mathbb{Z}_5$  there is a sector, called the  $k$ th Stokes sectors, whose asymptotic rays have arguments  $\varphi_{k-1}$  and  $\varphi_k$ . This sector will be denoted  $\Sigma_k$ . The boundary  $\partial \Sigma_k$  of each  $\Sigma_k$  is connected.*

*Any other sector has asymptotic rays with arguments  $\varphi_{k-1}$  and  $\varphi_{k+1}$ , for some  $k$ . We call such a sector the  $k$ th sector of band type, and we denote it  $B_k$ . The boundary  $\partial B_k$  of each  $B_k$  has two connected components.*

Choose a sector and a point  $\lambda_0$  belonging to it. The function  $S(\lambda_0, \lambda)$  maps a Stokes sector bi-holomorphically into the half plane and a  $B_k$  sector bi-holomorphically into a vertical strip.

**Definition 9.** *We call a differentiable curve  $\gamma : [0, 1] \rightarrow \mathbb{C}$  an admissible path provided  $\gamma$  is injective on  $[0, 1[$ ,  $\lambda_i \notin \gamma([0, 1])$ , for all turning points  $\lambda_i$ , and  $\text{Re}S(\gamma(0), \gamma(t))$  is a monotone function of  $t \in [0, 1]$ .*

*We say that  $\Sigma_j \rightleftharpoons \Sigma_k$  if there exist  $\mu_j \in \Sigma_j$ ,  $\mu_k \in \Sigma_k$  and an admissible path such that  $\gamma(0) = \mu_j$ ,  $\gamma(1) = \mu_k$ .*

The relation  $\rightleftharpoons$  is obviously reflexive and symmetric but it is not in general transitive.

Note that  $\Sigma_j \rightleftharpoons \Sigma_k$  if and only if for every point  $\mu_j \in \Sigma_j$  and every point  $\mu_k \in \Sigma_k$  an admissible path exists.

**Lemma 8.** *The relation  $\rightleftharpoons$  depends only on the equivalence class of the Stokes complex  $S$ .*

**Proof.** Consider an admissible path from  $\Sigma_j$  to  $\Sigma_k$ ,  $j \neq k$ . The path is naturally associated to the sequence of Stokes lines that it crosses. We denote the sequence  $l_n$ ,  $n = 0, \dots, N$ , for some  $N \in \mathbb{N}$ . We continue analytically  $S(\mu_j, \cdot)$  to a covering of the union of the Stokes sectors crossed by the path together with the Stokes lines belonging to the sequence. Since  $S(\mu_j, \cdot)$  is constant along each connected component of the boundary of every lift of a sector crossed by the path, then each of such connected components cannot be crossed twice by the path. Hence, due to the classification theorem no admissible path is a loop. Therefore, the union of the Stokes sectors crossed by the path together with the Stokes lines belonging to the sequence is simply connected.

Conversely, given any injective sequence of Stokes lines  $l_n$ ,  $n = 0 \dots, N$  such that for any  $0 \leq n \leq N - 1$ ,  $l_n$  and  $l_{n+1}$  belong to two different connected components of the boundary of a same sector, there exists an admissible path with that associated sequence. This last observation implies that the relation  $\rightleftharpoons$  depends only on the topology of the graph  $S$ . Moreover, if the sequence exists it is unique; indeed, if there existed two admissible paths,

**Table 1.** Computation of the relation  $\Leftrightarrow$ .

Graph	Pairs of non-consecutive sectors not satisfying the relation $\Leftrightarrow$
3 0 0	None
3 1 0	$(\Sigma_0, \Sigma_2), (\Sigma_0, \Sigma_{-2})$
3 1 1	$(\Sigma_1, \Sigma_{-1})$
3 2 0	$(\Sigma_1, \Sigma_{-1}), (\Sigma_1, \Sigma_{-2}), (\Sigma_{-1}, \Sigma_2)$
1 0 0	$(\Sigma_1, \Sigma_{-1}), (\Sigma_0, \Sigma_{-2}), (\Sigma_0, \Sigma_2)$
1 1 0	All but $(\Sigma_1, \Sigma_{-1})$
0 0 0	All

joining the same  $\mu_j$  and  $\mu_k$  but with different sequences, then there would be an admissible loop.  $\square$

With the help of lemma 8 and of the classification theorem, relation  $\Leftrightarrow$  can be easily computed, as is shown in table 1. As is evident from figure 1, for any graph type we have that  $\Sigma_k \Leftrightarrow \Sigma_{k+1}, \forall k \in \mathbb{Z}_5$ .

#### 4. Complex WKB method and asymptotic values

In this section we introduce the *WKB functions*  $j_k, k \in \mathbb{Z}_5$  and use them to evaluate the asymptotic values of equation (8). The topology of the Stokes complex will show all its importance in these computations.

On any Stokes sector  $\Sigma_k$ , we define the functions

$$S_k(\lambda) = S(\lambda^*, \lambda), \tag{16}$$

$$L_k(\lambda) = -\frac{1}{4} \int_{\lambda^*}^{\lambda} \frac{V'(u)}{V(u)} du, \tag{17}$$

$$j_k(\lambda) = e^{-S_k(\lambda)+L_k(\lambda)}. \tag{18}$$

Here  $\lambda^*$  is an arbitrary point belonging to  $\Sigma_k$  and the branch of  $\sqrt{V}$  is such that  $\text{Re}S_k(\lambda)$  is bounded from below.

We call  $j_k$  the *kth WKB function*.

##### 4.1. Maximal domains

In this subsection we construct the *kth maximal domain*, that we denote  $D_k$ . This is the domain of the complex plane where the *kth* WKB function approximates a solution of equation (8).

The construction is done for any  $k$  in a few steps (see figure 2 for the example of the Stokes complex of type (3 0 0)):

- (i) for every  $\Sigma_l$  such that  $\Sigma_l \Leftrightarrow \Sigma_k$ , denote  $D_{k,l}$  the union of the sectors and of the Stokes lines crossed by any admissible path connecting  $\Sigma_l$  and  $\Sigma_k$ .
- (ii) Let  $\widehat{D}_k = \cup_l D_{l,k}$ . Hence  $\widehat{D}_k$  is a connected and simply connected subset of the complex plane whose boundary  $\partial \widehat{D}_k$  is the union of some Stokes lines.
- (iii) Remove a  $\delta$ -tubular neighborhood of the boundary  $\partial \widehat{D}_k$ , for an arbitrarily small  $\delta > 0$ , such that the resulting domain is still connected.

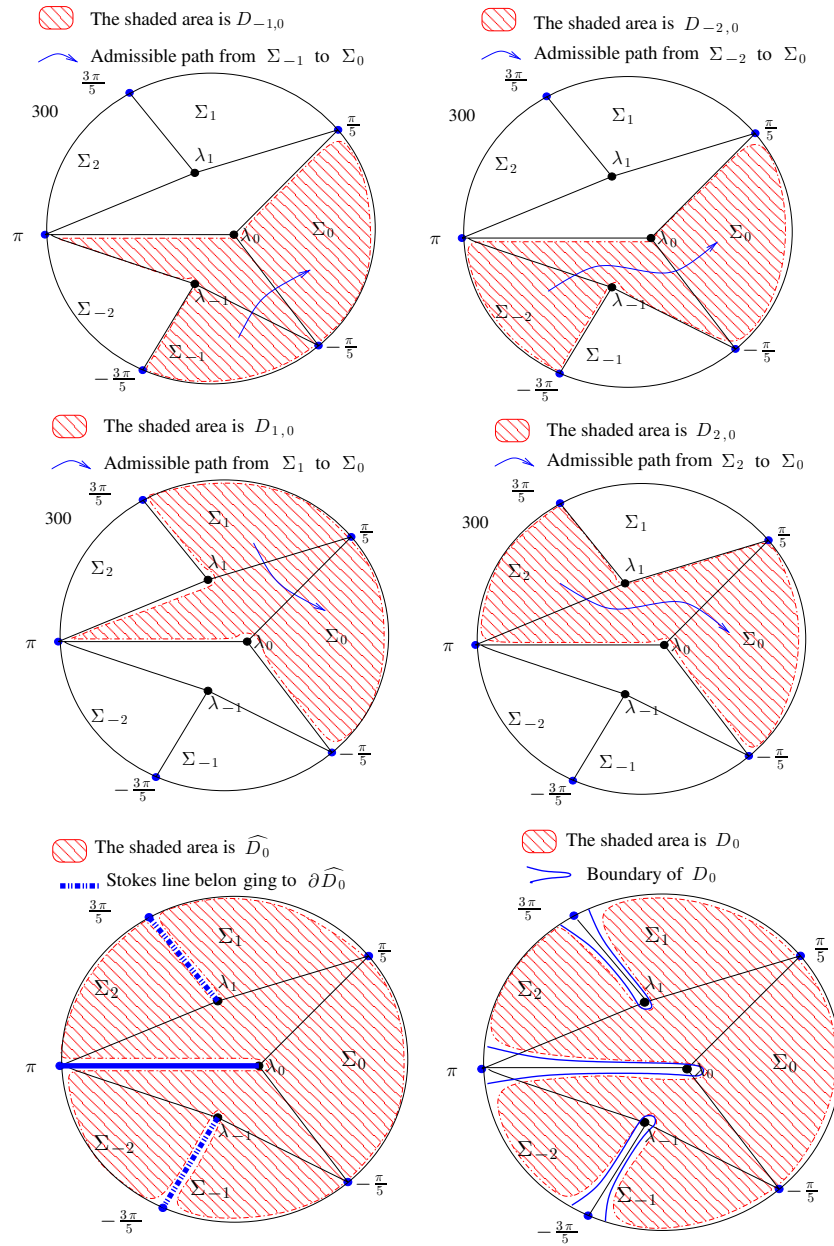


Figure 2. In the drawings, the construction of  $D_0$  for a graph of type (3 0 0) is depicted.

- (iv) For all  $l \neq k, l \neq k - 1$ , remove from  $\widehat{D}_k$  an angle  $\lambda = \rho e^{i\varphi}$ ,  $|\varphi - \varphi_l| < \epsilon, \rho > R$ , for  $\epsilon$  arbitrarily small and  $R$  arbitrarily big, in such a way that the resulting domain is still connected. The remaining domain is  $D_k$ .

#### 4.2. Main theorem of WKB approximation

We can now state the main theorem of the WKB approximation.

**Theorem 8** (G.D. Birkhoff [Bir33], Olver [Olv74]). *Continue the WKB function  $j_k$  to  $D_k$ . Then there exists a solution  $\psi_k(\lambda)$  of (8), such that for all  $\lambda \in D_k$*

$$\left| \frac{\psi_k(\lambda)}{j_k(\lambda)} - 1 \right| \leq g(\lambda)(e^{2\rho(\lambda)} - 1).$$

Here  $\rho_k$  is a bounded positive continuous function, called the error function, satisfying

$$\lim_{\substack{\lambda \rightarrow \infty \\ \varphi_{k-1} < \arg \lambda < \varphi_{k+1}}} \rho_k(\lambda) = 0,$$

and  $g(\lambda)$  is a positive function such that  $g(\lambda) \leq 1$  and

$$\lim_{\substack{\lambda \rightarrow \infty \\ \lambda \in D_k \cap \Sigma_{k \pm 2}}} g(\lambda) = \frac{1}{2}.$$

**Proof.** The proof is in the appendix A. □

Note that  $j_k$  is *sub-dominant* (i.e. it decays exponentially) in  $\Sigma_k$  and *dominant* (i.e. it grows exponentially) in  $\Sigma_l, \forall l \neq k$ .

For the properties of the error function,  $\psi_k$  is subdominant in  $\Sigma_k$  and dominant in  $\Sigma_{k \pm 1}$ .

### 4.3. Computations of asymptotic values in WKB approximation

The aim of this paragraph is to compute the asymptotic values for the Schrödinger equation (8) in WKB approximation. We explicitly work out the example of the Stokes complex of type (3 2 0), relevant to the study of poles of the intégrale tritronquée.

**Definition 10.** *Define the relative errors*

$$\rho_l^k = \begin{cases} \lim_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Sigma_k \cap D_l}} \rho_l(\lambda), & \text{if } \Sigma_l \rightleftharpoons \Sigma_k \\ \infty, & \text{otherwise} \end{cases}$$

and the asymptotic values

$$w_k(l, m) \stackrel{\text{def}}{=} w_k(\psi_l, \psi_m). \tag{19}$$

We say that  $\Sigma_k \sim \Sigma_l$  provided  $\rho_l^k < \frac{\log 3}{2}$ . The relation  $\sim$  is a sub-relation of  $\rightleftharpoons$ .

Note that  $\rho_l^{l+1} = 0$  and  $\rho_l^m = \rho_m^l$  (see appendix A).

In order to compute the asymptotic value  $w_k(l, m)$ , we have to know the asymptotic behavior of  $\psi_l$  and  $\psi_m$  in  $\Sigma_k$ . By theorem 8,

$$\lim_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Sigma_k \cap D_l}} \frac{\psi_l(\lambda)}{j_l(\lambda)} \neq 0, \quad \text{if } \frac{1}{2}(e^{2\rho_k^l} - 1) < 1.$$

Hence the asymptotic behavior of  $\psi_l$  in  $\Sigma_k$  can be related to the asymptotic behavior of  $j_l$  in  $\Sigma_k$  if the relative error  $\rho_l^k$  is so small that the above inequality holds true, i.e. if  $\Sigma_k \sim \Sigma_l$ .

**Remark.** Depending on the type of the graph  $S$ , there may not exist two indices  $k \neq l$  such that all the relative errors  $\rho_l^n, \rho_k^n, n \in \mathbb{Z}_5$  are small. However, it is often possible to compute an approximation of all the asymptotic values  $w_n(l, k)$  using the strategy below.

- (i) We select a pair of non consecutive Stokes sectors  $\Sigma_l, \Sigma_{l+2}$ , with the hypothesis that the functions  $\psi_l$  and  $\psi_{l+2}$  are linearly independent, so that  $w_l(l, l+2) = 0, w_{l+2}(l, l+2) = \infty$ . Since  $\rho_l^{l+1} = \rho_{l+2}^{l+1} = 0$  then

$$w_{l+1}(l, l+2) = \lim_{\lambda \in \Sigma_{l+1} \cap D_l \cap D_{l+2}, \lambda \rightarrow \infty} \frac{j_l(\lambda)}{j_{l+2}(\lambda)}.$$

Therefore, we find three exact and distinct asymptotic values.

- (ii) For any  $k \neq l, l+1, l+2$  such that  $\Sigma_l \sim \Sigma_k$  and  $\Sigma_{l+2} \sim \Sigma_k$ , we define the approximate asymptotic value

$$\widehat{w}_k(l, m) = \lim_{\lambda \in \Sigma_k \cap D_l \cap D_{l+2}, \lambda \rightarrow \infty} \frac{j_l(z)}{j_m(z)}.$$

The spherical distance between  $w_k(l, m)$  and  $\widehat{w}_k(l, m)$  may be easily estimated from above knowing the relative errors  $\rho_k^l$  and  $\rho_k^{l+2}$ .

If for any  $k \neq l, l+1, l+2, \Sigma_l \sim \Sigma_k$  and  $\Sigma_{l+2} \sim \Sigma_k$ , then the calculation is completed.

- (iii) If not, we can use the fact that quintuplets of asymptotic values for different choices of  $\psi_l, \psi_{l+2}$  are related by a Möbius transformation (see formula (12)). If for some pair  $(l, l+2)$  the assumption  $\Sigma_l \sim \Sigma_k, \Sigma_{l+1} \sim \Sigma_k$  fails to be true for just one value of the index  $k = k^*$ , and, for another pair  $(l', l'+2)$ , the assumption  $\Sigma_{l'} \sim \Sigma_{k'}, \Sigma_{l'+2} \sim \Sigma_{k'}$  fails to be true for just one valued of the index  $k' = k'^*$ , with  $k'^* \neq k^*$ , then there are three values of the index  $m \in \mathbb{Z}_5$  such that an approximation of  $w_m(l, l+2)$  and  $w_m(l', l'+2)$  is computable. Since any Möbius transformation is fixed by the action on three values, then we can compute an approximation of the transformation relating the quintuplets  $w_k(l, l+2)$  and  $w_k(l', l'+2)$  for any  $k \in \mathbb{Z}_5$ . Hence, we can calculate an approximation of the whole quintuplets  $w_k(l, l+2)$  and  $w_{k'}(l', l'+2)$ .

**Remark.** As shown in table 1, the relation  $\Leftrightarrow$  is uniquely characterized by the graph type. For the sake of computing the asymptotic values the important relation is  $\sim$  and not  $\Leftrightarrow$ . Indeed, the calculations for a given graph type, say  $(a b c)$ , are valid for (and only for) all the potentials whose relation  $\sim$  is equivalent to the relation  $\Leftrightarrow$  characterizing the graph type  $(a b c)$ .

Due to the above remark, in what follows we suppose that the relation  $\sim$  is equivalent to the relation  $\Leftrightarrow$ . We have the following.

**Lemma 9.** *Let  $V(\lambda; a, b)$  such that the type of the Stokes complex is  $(3 0 0), (3 1 0), (3 1 1)$ ; moreover, suppose that the  $\sim$  relation coincides with  $\Leftrightarrow$ . Then all the asymptotic values of equation (8) are pairwise distinct, with the possible exception of a single pair.*

**Proof.** For a graph of type  $(3 0 0)$  or  $(3 1 1)$  the thesis is trivial. For a graph of type  $(3 2 0)$ , it may be that  $w_0 = w_2$  or  $w_0 = w_{-2}$ . Since  $w_2 \neq w_{-2}$  the thesis follows.  $\square$

We completely work out the case of Stokes complex of type  $(3 2 0)$ , while for the other cases we present the results only. Due to lemma 9, we omit the results for potentials whose graph type is  $(3 0 0), (3 1 0)$  and  $(3 1 1)$ .

*Boutroux graph = 3 2 0.* We suppose that  $\Sigma_0 \sim \Sigma_{\pm 2}$ .

Let us consider first the pair  $\Sigma_0$  and  $\Sigma_{-2}$ . In figure 3 the maximal domains  $D_0$  and  $D_{-2}$  are depicted by coloring the Stokes lines not belonging to them blue and red, respectively. In particular  $S_0, L_0, j_0$  (resp.  $S_{-2}, L_{-2}, j_{-2}$ ) can be extended to all  $D_0$  (resp.  $D_{-2}$ ) along any curve that does not intersect any blue (resp. red) Stokes line.

We fix a point  $\lambda^* \in \Sigma_0$  such that  $S_0(\lambda^*) = S_{-2}(\lambda^*) = L_0(\lambda^*) = L_{-2}(\lambda^*) = 0$ .



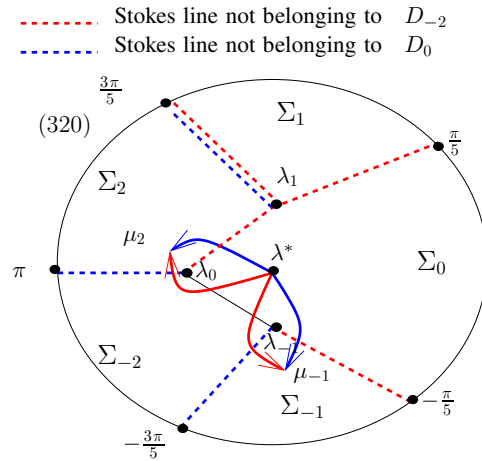


Figure 3. Calculation of  $w_{-1}(0, -2)$  and of  $\widehat{w}_2(0, -2)$

By definition

$$\begin{aligned} \widehat{w}_k(0, -2) &= \lim_{\lambda \rightarrow \infty_k} \frac{j_0(\lambda)}{j_{-2}(\lambda)} \\ &= \lim_{\lambda \rightarrow \infty_k} e^{-S_0(\lambda)+S_{-2}(\lambda)} e^{L_0(\lambda)-L_{-2}(\lambda)}, \end{aligned}$$

Here  $\lambda \rightarrow \infty_k$  is a short-hand notation for  $\lambda \rightarrow \infty, \lambda \in \Sigma_k \cap D_0 \cap D_{-2}$ . We calculate  $\widehat{w}_k(0, -2)$  for  $k = -1, 2$ .

We first calculate  $\lim_{\lambda \rightarrow \infty_k} e^{-S_0(\lambda)+S_{-2}(\lambda)}$ .

Note that  $\frac{\partial S_0}{\partial \lambda} = \frac{\partial S_{-2}}{\partial \lambda}$  in  $\Sigma_k$ . Hence,

$$\lim_{\lambda \rightarrow \infty_k} -S_0(\lambda) + S_{-2}(\lambda) = -S_0(\mu_k) + S_{-2}(\mu_k), \quad k = -1, 2,$$

where  $\mu_k$  is any point belonging to  $\Sigma_k$  (in figure 3, the paths of integration defining  $S_0(\mu_k)$  and  $S_{-2}(\mu_k)$  are colored blue and red, respectively).

On the other hand, since  $\frac{\partial S_0}{\partial \lambda} = -\frac{\partial S_{-2}}{\partial \lambda}$  in  $\Sigma_0 \cup \Sigma_{-2}$ , we have that

$$-S_0(\mu_k) + S_{-2}(\mu_k) = -2S_0(\lambda_s), \quad s = -1 \text{ if } k = -1 \text{ and } s = 0 \text{ if } k = 2.$$

We now compute  $\lim_{\lambda \rightarrow \infty_k} e^{L_0(\lambda)-L_{-2}(\lambda)}$ . Since  $\frac{\partial L_0}{\partial \lambda} = \frac{\partial L_{-2}}{\partial \lambda}$  in  $D_0 \cap D_{-2}$ , we have that

$$\lim_{\lambda \rightarrow \infty_k} L_0(\lambda) - L_{-2}(\lambda) = L_0(\mu_k) - L_{-2}(\mu_k),$$

$$L_0(\mu_k) - L_{-2}(\mu_k) = -\frac{1}{4} \oint_{c_k} \frac{V'(\mu)}{V(\mu)} d\mu, \quad k = -1, 2.$$

Here  $c_k$  is the blue path connecting  $\lambda^*$  with  $\mu_k$  composed with the inverse of the red path connecting  $\lambda^*$  with  $\mu_k$  (see figure 3).

Therefore, we have

$$\lim_{\lambda \rightarrow \infty_k} L_0(\lambda) - L_{-2}(\lambda) = -\sigma \frac{2\pi i}{4}, \quad \sigma = -1 \text{ if } k = -1 \text{ and } \sigma = +1 \text{ if } k = 2.$$

Combining the above computations, we get

$$w_{-1}(0, -2) = i e^{-2S_0(\lambda_{-1})}, \quad \widehat{w}_2(0, -2) = -i e^{-2S_0(\lambda_0)}.$$

We stress that  $w_{-1}(0, -2)$  is exact while  $\widehat{w}_2(0, -2)$  is an approximation.

Performing the same computations for the pair  $\Sigma_0$  and  $\Sigma_2$ , we obtain

$$w_1(0, 2) = -i e^{-2S_0(\lambda_1)}, \quad \widehat{w}_{-2}(0, 2) = i e^{-2S_0(\lambda_0)}.$$

Having calculated the triplet of asymptotic values  $w_0, w_2, w_{-2}$  for two different pairs of Stokes sectors, we can compute an approximation of the Möbius transformation relating all the asymptotic values for the two pairs:

$$\widehat{w}_k(0, -2) = -i e^{-2S_0(\lambda_0)} \frac{\widehat{w}_k(0, 2)}{\widehat{w}_k(0, 2) - i e^{-2S_0(\lambda_0)}}, \quad k \in \mathbb{Z}_5.$$

We eventually compute the last asymptotic value for the pair  $\Sigma_0, \Sigma_{-2}$ , that is

$$\widehat{w}_1(0, -2) = -i \frac{e^{-2S_0(\lambda_1)}}{1 + e^{-2(S_0(\lambda_1) - S_0(\lambda_0))}}.$$

*Quantization conditions.* The computations above provides us with the following quantization conditions:

$$\widehat{w}_1 = w_{-2} \Leftrightarrow e^{-2(S_0(\lambda_1) - S_0(\lambda_0))} = -1 \tag{20}$$

$$\widehat{w}_2 = w_{-1} \Leftrightarrow e^{-2(S_0(\lambda_{-1}) - S_0(\lambda_0))} = -1 \tag{21}$$

$$\widehat{w}_1 = w_{-1} \Leftrightarrow e^{-2(S_0(\lambda_1) - S_0(\lambda_{-1}))} = -1 + e^{-2(S_0(\lambda_1) - S_0(\lambda_0))}. \tag{22}$$

We note that equation (22) is incompatible both with (20) and (21). Equations (20) and (21) are Bohr–Sommerfeld quantizations.

As was shown in equation (13), the poles of the intégrale tritronquée are related to the polynomials such that  $w_1 = w_{-2}$  and  $w_{-1} = w_2$ . Since equations (20) and (21) can be simultaneously solved, solutions of system (20), (21) describe, in WKB approximation, polynomials related to the intégrale tritronquée. System (20), (21) was found by Boutroux in [Bou13] (through a completely different analysis), to characterize the asymptotic distribution of the poles of the intégrale tritronquée. Therefore, we call (20), (21) the *Bohr–Sommerfeld–Boutroux system*.

Equation (22) will not be studied in this paper, even though is quite remarkable. Indeed, it describes the breaking of the PT symmetry (see [DT00, BBM+01]).

*Case (1 0 0)*

$$\begin{aligned} w_0(1, -1) &= -1 \\ \widehat{w}_{-2}(1, -1) &= \widehat{w}_2(1, -1) = 1. \end{aligned}$$

Since  $w_0 \neq \widehat{w}_{\pm 2}$  and  $w_2 \neq w_{-2}$ , if the error  $\rho_1^{-2}$  or  $\rho_{-1}^2$  is small enough, then all the asymptotic values are pairwise distinct.

*Case (1 1 0)*

$$\begin{aligned} \widehat{w}_{-1}(1, -2) &= 1 \\ w_2(1, -2) &= -1. \end{aligned}$$

In this case, it is impossible to calculate  $w_0$  with the WKB method that has been here developed. Hence, it may be that either  $w_0 = w_2$  or  $w_0 = w_{-2}$ .

Note, however, that (1 1 0) is the graph only of a very restricted class of potentials namely  $V(\lambda) = (\lambda + \lambda_0)^2(\lambda - 2\lambda_0)$ , where  $\lambda_0$  is real and positive. Since the potential is real then  $w_0 \neq w_{\pm 2}$ .

*Case (0 0 0).* In this case, no asymptotic values can be calculated. Note, however, that  $V(\lambda) = \lambda^3$  is the only potential with graph (0 0 0). For this potential the asymptotic values

can be computed exactly, simply using symmetry considerations. Indeed one can choose  $w_k = e^{\frac{2k\pi}{5}i}$ ,  $k \in \mathbb{Z}_5$ .

#### 4.4. Small parameter

The WKB method normally applies to problem with an *external small parameter*, usually denoted  $\hbar$  or  $\varepsilon$ . In the study of the distributions of poles of a given solution  $y$  of P-I there is no external small parameter and we have to explore the whole space of cubic potentials. The aim of this section is to introduce a tool that allows us to investigate the validity of the WKB approximation.

On the linear space of cubic potentials in canonical form

$$V(\lambda; a, b) = 4\lambda^3 - 2a\lambda - 28b,$$

we define the following action of the group  $\mathbb{R}^+ \times \mathbb{Z}_5$  (similar to what is called Symanzik rescaling in [Sim70]):

$$(x, m)[V(\lambda; a, b)] = V(\lambda; \Omega^{2m} x^2 a, \Omega^{3m} x^3 b), \quad x \in \mathbb{R}^+, \quad m \in \mathbb{Z}_5, \quad \Omega = e^{\frac{2\pi}{5}i}. \quad (23)$$

The induced action on the graph  $S$ , on the relative error  $\rho_l^m$ , and on the difference  $S_i(\lambda_j) - S_i(\lambda_k)$  is described in the following.

**Lemma 10.** *Let the action of the group  $\mathbb{R}^+ \times \mathbb{Z}_5$  be defined as above. Then*

- (i)  $(x, m)$  leaves the graph  $S$  invariant, but for a shift of the labels  $k \rightarrow k + m$  of the external vertices.
- (ii)  $(x, m)[S_i(\lambda_j) - S_i(\lambda_k)] = x^{\frac{5}{2}}(S_i(\lambda_j) - S_i(\lambda_k))$ .
- (iii)  $(x, m)[\rho_l^k] = x^{-\frac{5}{2}}\rho_l^k$ .

**Proof.** The proof of (i) and (ii) follows from the following equality:

$$\sqrt{V(\lambda; \Omega^{2k} x^2 a, \Omega^{3k} x^3 b)} d\lambda = x^{\frac{5}{2}} \sqrt{V(\lambda'; a, b)} d\lambda', \quad \lambda = x\lambda'.$$

The proof of point (iii) follows from a similar scaling law of the 1-form  $\alpha(\lambda) d\lambda$  (see equation (A.5) in appendix A). □

Due to lemma 10 (iii),  $\varepsilon = \left| \frac{a}{b} \right|$  plays the role of the small parameter. Indeed, along any orbit of the action of the group  $\mathbb{R} \times \mathbb{Z}_5$ , all the (finite) relative errors go to zero uniformly as  $\left| \frac{a}{b} \right| \rightarrow 0$ . The reader should note that fixed one orbit  $\varepsilon \rightarrow 0$  if and only if  $a \rightarrow \infty$ . Hence, with the help of the WKB method we can study the asymptotic distribution of poles of the integr ale tritronqu ee.

Since all the relevant information is encoded in the quotient of the space of cubic potentials with respect to the group action, we define the following change of variable:

$$v(a, b) = \frac{b}{a}, \quad \mu(a, b) = \frac{b^2}{a^3}. \quad (24)$$

The induced action on these coordinates is simple, namely

$$(x, m)[\mu(a, b)] = \mu(a, b) \quad \text{and} \quad (x, m)[v(a, b)] = \Omega^m x v(a, b).$$

Moreover, the orbit of the set  $\{(v, \mu) \in \mathbb{C}^2 \text{ s.t. } |v| = 1, |\arg v| < \frac{\pi}{5}, \mu \neq 0\}$  is a dense open subset of the space of cubic potentials.

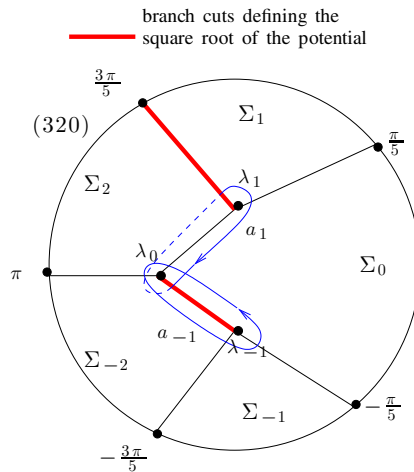


Figure 4. Riemann surface  $\mu^2 = V(\lambda; a, b)$

### 5. Poles of intégrale tritronquée

From lemma 9 and the results of the computations in section 4.3, it follows that equation (8) admits in WKB approximation the simultaneous solutions of the two quantization conditions  $w_{\pm 1} = w_{\mp 2}$  only if the Stokes complex is of type (3 2 0). In particular, after our calculations the poles of the intégrale tritronquée are related, in WKB approximation, to the solutions of the Bohr–Sommerfeld–Boutroux system (20) and (21).

We rewrite this system in the following equivalent form:

$$\begin{aligned} \oint_{a_1} \sqrt{V(\lambda; a, b)} d\lambda &= i\pi \left( n - \frac{1}{2} \right) \\ \oint_{a_{-1}} \sqrt{V(\lambda; a, b)} d\lambda &= -i\pi \left( m - \frac{1}{2} \right) \end{aligned} \tag{25}$$

where  $m, n$  are positive natural numbers and the paths of integration are shown in figure 4.

System (25) is studied in detail in [KK93] where the following lemma is proven.

**Lemma 11.** *If a polynomial  $V(\lambda) = 4\lambda^3 - 2a\lambda - 28b$  satisfies the system (25) then  $|\arg a| > \frac{4\pi}{5}$ .*

The Lemma above should be compared to the conjecture, to which we referred in the introduction.

*Real poles.* We compute all the real solutions of system (25) and compare them with some numerical results from [JK01]. We observe that the accuracy of the WKB method is astonishing also for small  $a$  and  $b$  (see table 2 below). Such accuracy cannot be explained by means of the tools developed in the present paper and it will be investigated in a subsequent publication of the author.

For the subset of real potentials, we have

$$\oint_{a_1} \sqrt{V(\lambda; a, b)} d\lambda = \oint_{a_{-1}} \sqrt{V(\lambda; a, b)} d\lambda,$$

**Table 2.** Comparison between numerical and WKB evaluation of the first two real poles of the intégrale tritronquée.

	WKB	Numeric	Error %
$a_1$	-2, 34	-2, 38	1, 5
$b_1$	-0, 064	-0, 062	2
$\mu_1$	-3158	-3510	10
$a_2$	-5, 65	-5, 66	0, 2
$b_2$	-0, 23	unknown	unknown
$\mu_2$	-3158	unknown	unknown

where  $\bar{\phantom{x}}$  stands for complex conjugation.

Therefore system (25) reduces to one equation and the real poles of tritronquée are characterized, in WKB approximation, by one natural number.

**Lemma 12.** *Let  $\mu$  and  $\nu$  be defined as in equation (24). Then the real polynomials whose Stokes complex is of type (3 2 0) are the orbit of a single point of the  $\mathbb{R}^+$  action, characterized by  $\mu^* \cong -3158,92$  and  $\nu > 0$ .*

*Moreover, if a real polynomial  $V(\lambda; a, b)$  satisfies the Bohr–Sommerfeld–Boutroux conditions (25), then*

$$a = a^* \left(n - \frac{1}{2}\right)^{\frac{4}{5}}, \quad b = b^* \left(n - \frac{1}{2}\right)^{\frac{6}{5}} \tag{26}$$

for some  $n \in \mathbb{N}$  and  $a^* \cong -4,0874, b^* \cong -0,1470$ .

**Proof.** All real cubic potentials whose Stokes complex has type (3 2 0) have one real turning point  $\lambda_0$  and two complex conjugate turning points  $\lambda_{\pm 1}$ . For the subset of real potentials with two complex conjugate turning points the cycles  $a_{\pm 1}$ , as shown in figure 4, are unambiguously defined. From the classification theorem, it follows that a Stokes complex has type (3 2 0) if and only if  $\operatorname{Re} \int_{a_1} \sqrt{V(\lambda; a, b)} d\lambda = 0, \operatorname{Im} \int_{a_1} \sqrt{V(\lambda; a, b)} d\lambda \neq 0$ . Since these conditions are invariant under the  $\mathbb{R}^+$  action, if they are satisfied for a point of the space of cubic potentials, then they are satisfied on all its orbit. Moreover, it is easily seen that this orbit exists and is unique. With the help of software of numeric calculus, we characterized numerically the orbit. After that, using the scaling law in lemma 10 (ii) we calculated all the real solutions of the Bohr–Sommerfeld–Boutroux system.  $\square$

To the best of our knowledge, the asymptotic for the  $b$  coefficients has never been given.

In the paper [JK01], the authors showed that the intégrale tritronquée has no poles on the real positive axis. The real poles are a decreasing sequence of negative numbers  $a_n$  and some of them are evaluated numerically in the same paper.

In table 2, we compare the first two real solutions to system (25) with the numerical evaluation of the first two poles of the intégrale tritronquée.

## 6. Concluding remarks

We have studied the distribution of the poles of solutions to the Painlevé first equation using the theory of the cubic anharmonic oscillator. We have applied a suitable version of the complex WKB method to analyze the distribution of poles of the intégrale tritronquée.

In subsequent publications we plan to pursue our study of poles of P-I transcendents in different directions.

In particular, we want to use the Nevanlinna theory of the branched coverings of the sphere to complete the analysis of the poles of the intégrale tronquée, showing that the developed WKB method yields a complete qualitative picture and efficient quantitative estimates of the distribution of the poles.

Since the quantization condition  $\sigma_0 = 0$  characterizes the monodromy data of a family of special solutions of P-I, called *intégrale tronquée* (see [Kap04]), these solutions are strictly related to the spectral theory of PT symmetric anharmonic oscillators and to functional equations of Bethe Ansatz type (for what concerns the PT symmetric anharmonic oscillators and the Bethe Ansatz equations, see [DDT01] and references therein). We will investigate the consequences of this relation in a subsequent publication.

**Remark.** After the main computations of the present paper had been completed, the author learned from B. Dubrovin about the results of V. Novokshenov presented at the conference NEEDS09 (May 2009). Novokshenov studied WKB solutions to the Schrödinger equation (8) with  $b = 0$  and their connections to the distributions of poles of certain particular solutions to the Painlevé-I equation, including intégrales tronquée and intégrale tronquée.

### Acknowledgments

I am indebted to Professor B Dubrovin who introduced me to the problem and constantly gave me suggestions and advice. I would like to thank A Raimondo, G De Nittis and R Tateo for useful discussions. I am grateful to the anonymous referee for useful suggestions.

Part of this work was written during my visit at the department ‘R Caccioppoli’ of Napoli University. This work is partially supported by the European Science Foundation Programme ‘Methods of Integrable Systems, Geometry, Applied Mathematics’ (MISGAM), and by the Italian Ministry of University and Research (MIUR) grant PRIN 2006 ‘Geometric methods in the theory of nonlinear waves and their applications’.

### Appendix A

The aim of this appendix is to prove theorem 8. Our approach is similar to the approach of Fedoryuk [Fed93].

Notations are as in sections 3 and 4, except for  $\infty_k$ . In what follows, we suppose to have fixed a certain cubic potential  $V(\lambda; a, b)$  and a maximal domain  $D_k$ . To simplify the notation we write  $V(\lambda)$  instead of  $V(\lambda; a, b)$ .

#### *Gauge transform to an L-diagonal system*

The strategy is to find a suitable gauge transform of equation (8) such that for large  $\lambda$  it simplifies. We rewrite the Schrödinger equation

$$-\psi''(\lambda) + V(\lambda)\psi(\lambda) = 0, \quad (\text{A.1})$$

in first-order form:

$$\Psi'(\lambda) = E(\lambda)\Psi(\lambda), \quad E(\lambda) = \begin{pmatrix} 0 & 1 \\ V(\lambda) & 0 \end{pmatrix}. \quad (\text{A.2})$$

**Lemma 13** [Fedoryuk]. *In  $D_k$*

(i) *the gauge transform*

$$Y(\lambda) = A(\lambda)U(\lambda),$$

$$A(\lambda) = j_k(\lambda) \begin{pmatrix} 1 & 1 \\ \sqrt{V(\lambda)} - \frac{V'(\lambda)}{4V(\lambda)} & -\sqrt{V(\lambda)} - \frac{V'(\lambda)}{4V(\lambda)} \end{pmatrix}, \quad (\text{A.3})$$

*is non-singular and*

(ii) *system (A.2) is transformed into the following one:*

$$U'(\lambda) = F(\lambda)U(\lambda) = (A^{-1}EA - A^{-1}A')U,$$

$$F(\lambda) = 2\sqrt{V(\lambda)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \alpha(z) \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad (\text{A.4})$$

$$\alpha(\lambda) = \frac{1}{32\sqrt{V(\lambda)}^5} (4V(\lambda)V''(\lambda) - 5V'^2(\lambda)). \quad (\text{A.5})$$

**Proof.**

- (i) Indeed  $\det A(\lambda) = 2j_k^2(\lambda)\sqrt{V(\lambda)} \neq 0, \forall \lambda \in D_k$ , by construction of  $j_k$  and  $D_k$ .
- (ii) It is proven by a simple calculation. □

*Some technical lemmas*

Before we can begin the proof of theorem 8, we have to introduce the compactification of  $D_k$  and the preparatory lemma 14.

*Compactification of  $D_k$ .* Since  $D_k$  is simply connected, it is conformally equivalent to the interior of the unit disk  $D$ . We denote  $U$  the uniformization map,  $U : D \rightarrow D_k$ .

By construction, the boundary of  $D_k$  is the union of  $n$  free Jordan curves, all intersecting at  $\infty$ . Here  $n$  is equal to the number of sectors  $\Sigma_l$  such that  $\Sigma_l \rightleftharpoons \Sigma_k$  minus 2.

Due to an extension of Carathéodory’s theorem ([Car], section 134–138), the map  $U$  extends to a continuous map from the closure of the unit circle to the closure of  $D_k$ . The map is injective on the closure of  $D$  minus the  $n$  counterimages of  $\infty$ . Hence, the uniformization map realizes a  $n$  point compactification of  $D_k$ , that we call  $\overline{D}_k$ . In  $\overline{D}_k$  there are  $n$  point at  $\infty$ . We denote  $\infty_k$  the point at  $\infty$  belonging to the closure of  $U(\Sigma_{k-1} \cup \Sigma_k \cup \Sigma_{k+1})$ . Moreover, for  $\lambda = k + 2$  or  $\lambda = k - 2$ , if  $\Sigma_l \rightleftharpoons \Sigma_k$  we denote  $\infty_l$  the point at  $\infty$  belonging to the closure of  $U(\Sigma_l)$ .

**Definition 11.** *Let  $H$  be the space of function holomorphic in  $D_k$  and continuous in  $\overline{D}_k$ .  $H$  endowed with the sup norm is a Banach space  $(H, \|\cdot\|_H)$ .*

*Let  $\Gamma(\lambda), \lambda \in \overline{D}_k - \infty_k$  be the set of injective piecewise differentiable curves  $\gamma : [0, 1] \rightarrow \overline{D}_k$ , such that*

- (1)  $\gamma(0) = \lambda, \gamma(1) = \infty_k$ ,
- (2)  $Re S_k(\gamma(0), \gamma(t))$  is eventually non-decreasing,
- (3) there is an  $\varepsilon > 0$  such that eventually  $|\arg \gamma(t) - \frac{2\pi k}{5}| < \frac{\pi}{5} - \varepsilon$ ,
- (4) the length of the curve restricted to  $[0, T]$  is  $O(|\gamma(T)|)$ , as  $T \rightarrow 1$ .

Let  $\tilde{\Gamma}(\lambda)$  be the subset of  $\Gamma(\lambda)$  of the paths along which  $ReS_k(\gamma(0), \gamma(t))$  is non-decreasing.

Let  $K_1 : H \rightarrow H$  and  $K_2 : H \rightarrow H$  be defined (for the moment formally)

$$K_1[h](\lambda) = - \int_{\gamma \in \Gamma(\lambda)} e^{2S_k(\mu, \lambda)} \alpha(\mu) h(\mu) d\mu, \tag{A.6}$$

$$K_2[h](\lambda) = \int_{\gamma \in \Gamma(\lambda)} \alpha(\mu) h(\mu) d\mu. \tag{A.7}$$

Let  $\rho : \overline{D_k} \rightarrow \overline{D_k}$ :

$$\rho(\lambda) = \begin{cases} \inf_{\gamma \in \tilde{\Gamma}(\lambda)} \int_0^1 |\alpha(\gamma(t)) \frac{d\gamma(t)}{dt}| dt, & \text{if } \lambda \neq \infty_k \\ 0, & \text{if } \lambda = \infty_k. \end{cases}$$

**Remark 8.** Since along rays of fixed argument  $\varphi$ , with  $|\varphi - \frac{2\pi k}{5}| < \frac{\pi}{5} - \varepsilon$ ,  $ReS_k$  is eventually increasing, there are paths satisfying point (1) through (4) of the above definition. Moreover, by construction of  $D_k$ ,  $\tilde{\Gamma}(\lambda)$  is non empty for any  $\lambda$ .

Before beginning the proof of the theorem, we need the following preparatory.

**Lemma 14.**

- (i)  $\rho$  is a continuous function.
- (ii)  $K_1$  and  $K_2$  are well-defined bounded operator. In particular

$$|K_i[h](\lambda)| \leq \rho(\lambda) \|h\|_H, \quad i = 1, 2 \tag{A.8}$$

- (iii)  $K_2[h](\infty_k) = K_1[h](\infty_k) = K_1(\infty_{k\pm 2}) = 0, \forall h \in H$ .

**Proof.** The proof is rather standard and left to the reader. □

We are now ready to prove theorem 8.

**Theorem 9.** Extend the WKB function  $j_k$  to  $D_k$ . There exists a unique solution  $\psi_k$  of (8) such that for all  $\lambda \in D_k$

$$\begin{aligned} \left| \frac{\psi_k(\lambda)}{j_k(\lambda)} - 1 \right| &\leq g(\lambda)(e^{2\rho(\lambda)} - 1) \\ \left| \frac{\psi'_k(\lambda)}{j_k(\lambda)\sqrt{V(\lambda)}} + 1 \right| &\leq \left| \frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}} \right| + \left( 1 + \left| \frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}} \right| \right) g(\lambda)(e^{2\rho(\lambda)} - 1), \end{aligned}$$

where  $g(\lambda)$  is a positive function,  $g(\lambda) \leq 1$  and  $g(\infty_{k\pm 2}) = \frac{1}{2}$ .

**Proof.** We seek a particular solution to the linear system (A.4) via successive approximation.

If  $U(\lambda) = U^{(1)} \oplus U^{(2)} \in H \oplus H$  satisfies the following integral equation of Volterra type:

$$\begin{aligned} U(\lambda) &= U_0 + K[U](\lambda), \quad U_0 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ K[U](\lambda) &= \begin{pmatrix} K_1[U^{(1)} + U^{(2)}](\lambda) \\ K_2[U^{(1)} + U^{(2)}](\lambda) \end{pmatrix}, \end{aligned} \tag{A.9}$$

then  $U(\lambda)$  restricted to  $D_k$  satisfies (A.4).



We define the *Neumann series* as follows:

$$U_{n+1} = U_0 + K[U_n], \quad U_{n+1} = \sum_{i=0}^{n+1} K^i[U^0]. \tag{A.10}$$

Due to lemma 14 and to general properties of Volterra integral operators, we find the following estimate:

$$|K^n[U_0]^{(i)}(\lambda)| \leq \frac{1}{2} \frac{1}{n!} (2\rho(\lambda))^n, \tag{A.11}$$

where  $K^n[U_0]^{(i)}$  is the second component of  $K^n[U_0]$ .

Thus, the sequence  $U^n$  converges in  $H$  and is a solution to (A.9); call  $U$  its limit. Due to lemma 14,  $U^{(1)}(\infty_{k\pm 2}) = 0$ .

Let  $\Psi_k$  be the solution to (A.2) whose gauge transform is  $U$  restricted to  $D_k$ ; The first component  $\psi_k$  of  $\Psi_k$  satisfies equation (A.1).

From the gauge transform (A.3), we obtain

$$\begin{aligned} \frac{\psi_k(\lambda)}{j_k(\lambda)} - 1 &= U_1(\lambda) + U_2(\lambda) - 1, \\ \frac{\psi'_k(\lambda)}{j_k(\lambda)\sqrt{V(\lambda)}} + 1 &= U_1(\lambda) \left(1 - \frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}}\right) - (U_2(\lambda) - 1) \left(1 + \frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}}\right) - \frac{V'(\lambda)}{4V(\lambda)^{\frac{3}{2}}}, \end{aligned}$$

The thesis follows from these formulas, inequality (A.11) and from the fact that  $U_1(\infty_{k\pm 2}) = 0$ . □

**Remark.** The solution  $\psi_k(\lambda)$  of equation (8) described in theorem 8 may be extended from  $D_k$  to the whole complex plane, since the equation is linear with entire coefficients. The continuation is constructed in the following corollary.

**Corollary 1.** For any  $\lambda \in \mathbb{C}$ ,  $\lambda$  not a turning point, we define  $\Gamma(\lambda)$  as in definition 11. Fixed any  $\gamma \in \Gamma(\lambda)$  and  $h$  a continuous function on  $\gamma$ , we define the functionals  $K_i[h](\lambda)$  as in equations (A.6) and (A.7). We define the Neumann series as in equations (A.9) and (A.10), and we continue  $j_k$  along  $\gamma$ .

Then Neumann series converges and we call  $U^{(1)}(\lambda)$  and  $U^{(2)}(\lambda)$  the first and second component of its limit.

Moreover,  $\psi_k(\lambda) = (U^{(1)}(\lambda) + U^{(2)}(\lambda))j_k(\lambda)$  solves equation (8) and for any  $\varepsilon > 0$

$$\lim_{|\lambda| \rightarrow \infty, |\arg \lambda - \frac{2\pi k}{5}| < \frac{3\pi}{5} - \varepsilon} (U^{(1)}(\lambda) + U^{(2)}(\lambda)) = 1.$$

The reader should note that if  $\lambda \notin D_k$ , then  $\tilde{\Gamma}(\lambda)$  is empty and we cannot estimate  $\frac{\psi_k(\lambda)}{j_k(\lambda)}$ .

## Appendix B

The aim of this appendix is to prove theorem 2 (ii) and (iii). The notation is, if not otherwise stated, as in the previous sections of the paper.

Next to a pole  $z = a$  of a solution  $y(z)$  of P-I, equation (1) becomes meaningless. To get rid of this singularity we perform a gauge transform of (1) such that the gauge-transformed equation has full meaning in the limit. In what follows, we suppose that  $z$  belongs to a punctured neighborhood of  $a$ , where  $y(z)$  is holomorphic.

A gauge transform

Let  $z$  be a fixed regular value of  $y(z)$ . Let  $\vec{\Phi}(\lambda, z) = G(\lambda, z)\vec{\Psi}(\lambda, z)$ ,

$$G(\lambda, z) = \begin{pmatrix} \frac{y'(z) + \frac{1}{2(\lambda - y(z))}}{\sqrt{2(\lambda - y(z))}} & \frac{1}{\sqrt{2(\lambda - y(z))}} \\ \sqrt{2(\lambda - y(z))} & 0 \end{pmatrix}. \tag{B.1}$$

Then  $\vec{\Phi}(\lambda; z)$  satisfies (1) if and only if  $\vec{\Psi}(\lambda; z)$  satisfies the following equation:

$$\Psi_\lambda(\lambda, z) = \begin{pmatrix} 0 & 1 \\ Q(\lambda; z) & 0 \end{pmatrix} \Psi(\lambda, z)$$

where

$$Q(\lambda; z) = 4\lambda^3 - 2\lambda z + 2zy(z) - 4y^3(z) + y^2(z) + \frac{y'(z)}{\lambda - y(z)} + \frac{3}{4(\lambda - y(z))^2}. \tag{B.2}$$

We denote  $\psi$  the first component of  $\vec{\Psi}$ . The equation for  $\vec{\Psi}$  is equivalent to the following second order scalar equation for  $\psi$ :

$$\psi_{\lambda\lambda}(\lambda, z) = Q(\lambda; z)\psi(\lambda, z) \tag{B.3}$$

We summarize some property of the perturbed potential, which can be easily verified using the expansion (7).

**Lemma 15.** Let  $\varepsilon^2 = \frac{1}{y(z)} = (z - a)^2 + O((z - a)^6)$ ; then

- (i)  $Q(\lambda; z)$  has a double pole at  $\lambda = \frac{1}{\varepsilon^2}$ . It is an apparent fuchsian singularity for equation (B.3): the local monodromy around it is  $-1$ .
- (ii)  $Q(\lambda; z)$  has two simple zeros at  $\lambda = \frac{1}{\varepsilon^2} + O(\varepsilon^2)$
- (iii)  $Q(\lambda; z) = 4\lambda^3 - 2(a + \varepsilon)\lambda - 28b + O(\varepsilon) - \frac{2\lambda\varepsilon^{-1}}{\lambda - \varepsilon^{-2}} + \frac{3}{4(\lambda - \varepsilon^{-2})^2}$ , where  $O(\varepsilon)$  does not depend on  $\lambda$ .

Equation (B.3) is a perturbation of the cubic Schrödinger equation (8) and the asymptotic behaviors of solutions to the two equations are very similar. Indeed the local picture around the point at  $\infty$  depends only on the terms  $4\lambda^3$  and  $-2z\lambda^3$ .

More precisely, the equivalent of corollary 1 in appendix A is valid also for the perturbed Schrödinger equation.

**Definition 12.** For any  $z$ , define a cut from  $\lambda = \frac{1}{(z-a)^2}$  to  $\infty$  such that it eventually does not belong to the the angular sector  $|\arg \lambda - \frac{2\pi k}{5}| \leq \frac{3\pi}{5}$ .

Fix  $\lambda^*$  in the cut plane.  $S_k(\lambda; z) = \int_{\lambda^*}^{\lambda} \sqrt{Q(\mu; z)} d\mu$  is well-defined for  $|\arg \lambda - \frac{2k\pi}{5}| < \frac{3\pi}{5}$  and  $\lambda \gg 0$ . Here the branch of  $\sqrt{Q}$  is chosen such that  $Re S_k(\lambda) \rightarrow +\infty$  as  $|\lambda| \rightarrow \infty$ ,  $|\arg \lambda - \frac{2\pi k}{5}| < \frac{\pi}{5} - \varepsilon$ . We define  $j_k(\lambda; z)$  as in equation (18) and  $\alpha(\lambda; z)$  as in equation (A.5), but replacing  $V(\lambda)$  with  $Q(\lambda; z)$ .

For any  $\lambda$  in the cut plane, let  $\Gamma(\lambda)$  be the set of piecewise differentiable curves  $\gamma : [0, 1]$  to the cut plane,  $\gamma(0) = \lambda$ ,  $\gamma(1) = \infty$ , satisfying properties (2), (3) and (4) of definition 11.

For any  $\gamma \in \Gamma(\lambda)$ , let  $H$  be the Banach space of continuous functions on  $\gamma$  that have a finite limit as  $t \rightarrow 1$ . Formulas (A.6) and (A.7) define two bounded functionals on  $H$ . We call such functionals  $K_1(\lambda; z)$  and  $K_2(\lambda; z)$ .

Following the proof of theorem 8, the reader can prove the following.

**Lemma 16.** Let  $\lambda$  belong to the cut plane,  $\lambda$  not a zero of  $Q(\cdot; z)$ . Fixed any  $\gamma \in \Gamma(\lambda)$ , we define the Neumann series as in equations (A.9) and (A.10), and we continue  $j_k$  along  $\gamma$ .

Then the Neumann series converges and  $\psi_k(\lambda) = (U_1(\lambda) + U_2(\lambda)) j_k(\lambda)$  solves equation (B.3). Moreover, for any  $\varepsilon > 0$

$$\lim_{|\lambda| \rightarrow \infty, |\arg \lambda - \frac{2\pi k}{5}| < \frac{3\pi}{5} - \varepsilon} (U^{(1)}(\lambda) + U^{(2)}(\lambda)) = 1$$

**Definition 13.** Let  $\tilde{\psi}_k(\lambda, z)$  be the unique solution of equation (B.3) such that

$$\frac{\tilde{\psi}_k(\lambda, z)}{\lambda^{-\frac{3}{4}} e^{-\frac{4}{5} \lambda^{\frac{5}{2}} + z \lambda^{\frac{1}{2}}}} \rightarrow 1, \quad \text{as } |\lambda| \rightarrow \infty, \quad \arg \lambda = \frac{2\pi k}{5}. \quad (\text{B.4})$$

Here the branch of  $\lambda^{\frac{1}{4}}$  is fixed as  $\lambda \rightarrow \infty$ ,  $\arg \lambda = \frac{2\pi k}{5}$ , and there it coincides with the branch chosen in equation (3). We define  $\tilde{\psi}_k(\lambda, a)$  to be the unique solution of equation (8) with asymptotic (B.4), where  $z = a$ .

We denote  $\psi_k(\lambda; z)$  the unordered pair  $\{\tilde{\psi}_k(\lambda, z), -\tilde{\psi}_k(\lambda, z)\}$ .

**Remark.** We note that if the cuts are continuous in  $z$ , then  $\tilde{\psi}_k(\lambda, z) = c(z)\psi_k(\lambda)$ , where  $\psi_k(\lambda)$  is the solution constructed in lemma 16 and  $c(z)$  is a bounded holomorphic function.

**Theorem 10.**  $\lim_{z \rightarrow a} \psi_k(\lambda, z) = \psi_k(\lambda, a), \forall \lambda \in \mathbb{C}$ .

**Proof.** Let  $\lambda$  be any point in the complex plane which is not a zero of  $V(\lambda; a, b)$ . For any sequence  $\varepsilon_n$  converging to zero, we choose two fixed rays  $r_1$  and  $r_2$  of different argument  $\varphi_1$  and  $\varphi_2$ ,  $|\varphi_i - \frac{2k\pi}{5}| < \frac{\pi}{5}$ . We denote  $D_{R,\varepsilon}$  a disk of radius  $R$  with center  $\lambda = \frac{1}{\varepsilon^2}$  and we split the sequence  $\varepsilon_n$  into two subsequences  $\varepsilon_n^i$  such that  $r_i \cap D_{R,\varepsilon_n^i} = \emptyset$  for any  $n$  big enough.

For  $i = 1, 2$ , we choose the cuts defined in definition 12 in such a way that there exists a differentiable curve  $\gamma_i : [0, 1] \rightarrow \mathbb{C}$ ,  $\gamma_i(0) = \lambda$ ,  $\gamma_i(1) = \infty$  with the following properties: (i)  $\gamma_i$  avoids the zeroes of  $Q(\lambda, \varepsilon_n^i)$  and a fixed, arbitrarily small, neighborhood of the zeroes of  $V(\lambda; a, b)$ , (ii)  $\gamma_i$  does not intersect any cut, and (iv)  $\gamma_i$  eventually lies on  $r_i$ .

The proof of the thesis relies on the following estimates:

$$\begin{aligned} \sup_{\lambda \in \mathbb{C} - D_{R,\varepsilon}} |\lambda^{-\delta}| |Q(\lambda; a + \varepsilon) - V(\lambda; a, b)| &= O(\varepsilon^{2\delta-3}), \\ \sup_{\lambda \in \mathbb{C} - D_{R,\varepsilon}} |\lambda^{-\delta}| |Q_\lambda(\lambda; a + \varepsilon) - V_\lambda(\lambda; a, b)| &= O(\varepsilon^{2\delta-3}), \\ \sup_{\lambda \in \mathbb{C} - D_{R,\varepsilon}} |\lambda^{-\delta}| |Q_{\lambda,\lambda}(\lambda; a + \varepsilon) - V_{\lambda,\lambda}(\lambda; a, b)| &= O(\varepsilon^{2\delta-3}). \end{aligned} \quad (\text{B.5})$$

Due the above estimates it is easily seen that  $\gamma_i \in \Gamma(\lambda), \forall \varepsilon_n^i$ . Due to lemma 16 and corollary 1, to prove the thesis it is sufficient to show that the functionals  $K_1(\lambda; a + \varepsilon_n^i)$  and  $K_2(\lambda; a + \varepsilon_n^i)$  converge in norm to  $K_1(\lambda; a)$  and  $K_2(\lambda; a)$ . Here  $K_i(\lambda; a), i = 1, 2$  are defined as in corollary 1. We note that the norm of the functionals are just the  $L^1(\gamma_i)$  norm of their integral kernels.

We first consider the functionals  $K_2(\lambda; a + \varepsilon_n^i)$ . Due to the above estimates

$$\lambda^{\frac{7}{2}} \alpha(\mu, \varepsilon_n^i) \rightarrow \lambda^{\frac{7}{2}} \alpha(\mu), \text{ uniformly on } \gamma_i([0, 1]) \quad \text{as } n \rightarrow \infty.$$

Hence the sequence  $\alpha(\mu, \varepsilon_n^i)$  converges in norm  $L^1(\gamma_i)$  to  $\alpha(\mu)$  and the sequence  $K_2(\lambda; \varepsilon_n^i)$  converges in operator norm to  $K_2(\lambda; a)$ .

We consider now the sequence  $K_1(\lambda; a + \varepsilon_n^i)$ .

To prove the convergence of the above sequence of operators, it is sufficient to prove that

$$e^{S_k(\lambda; a + \varepsilon_n^i) - S_k(\mu; a + \varepsilon_n^i)} \rightarrow e^{S_k(\lambda; 0) - S_k(\mu; 0)} \text{ uniformly on } \gamma_i([0, 1]) \quad \text{as } n \rightarrow \infty.$$

We first note that

$$e^{S_k(\lambda;0)-S_k(\mu;0)} - e^{S_k(\lambda;a+\varepsilon_n^i)-S_k(\mu;a+\varepsilon_n^i)} = e^{S_k(\lambda;0)-S_k(\mu;0)}(1 - e^{g(\mu;\varepsilon)}),$$

$$g(\mu, \varepsilon) = \int_{\lambda, \gamma_i}^{\mu} \frac{Q(v, \varepsilon) - V(v; a, b)}{\sqrt{Q(v, \varepsilon)} + \sqrt{P(v; a, b)}} dv.$$

Using estimate (B.5), it is easy to show that  $g(\mu; \varepsilon) = f(\varepsilon)O(\mu^\delta)$ , where  $f(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and  $0 < \delta \ll 1$ . Therefore, the difference of the exponential functions converges uniformly to 0.  $\square$

We can prove theorem 2 (ii) and (iii).

Indeed from (A.3), it is easily seen that (choosing one of the two branches of the gauge transform)

$$\vec{\Psi}_k(\lambda; z) = G(\lambda, z)^{-1} \vec{\Phi}_k(\lambda; z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{\psi}_k(\lambda, z) \\ \tilde{\psi}'_k(\lambda, z) \end{pmatrix},$$

if  $|\arg \lambda - \frac{2\pi k}{5}| < \frac{3\pi}{5}$  and  $|\lambda| \gg 0$ .

Moreover from (A.3), it follows that

$$\lim_{z \rightarrow a} (z - a) \Phi_k^{(2)}(\lambda, z) = i\sqrt{2} \Psi_k^{(1)}(\lambda; a).$$

Hence, theorem 2 (ii) and (iii) follow from theorem 10.

## References

- [BBM+01] Bender C M, Berry M, Meisinger P N, Savage V M and Simsek M 2001 Complex WKB analysis of energy-level degeneracies of non-hermitian hamiltonians *J. Phys. A: Math. Gen.* **34** L31
- [Bir33] Birkhoff G D 1933 Quantum mechanics and asymptotic series *Bull. Am. Math. Soc.* **39** 681–700
- [Bou13] Boutroux P 1913 Recherches sur les transcendentes de M. Painlevé et l'étude asymptotique des équations différentielles du second ordre *Ann. École Norm.* **30** 255–375
- [BW68] Bender C and Wu T 1968 Anharmonic oscillator *Phys. Rev. (2)* **184** 1231–60
- [Car] Carathéodory C *Conformal Representation* 2nd edn (Cambridge: Cambridge University Press)
- [DDT01] Dorey P, Dunning C and Tateo R 2001 Spectral equivalences, Bethe ansatz equation, and reality properties in PT-symmetric quantum mechanics *J. Phys. A: Math. Gen.* **34** 5679
- [DGK09] Dubrovin B, Grava T. and Klein C. 2009 On universality of critical behaviour in the focusing nonlinear Schrödinger equation, elliptic umbilic catastrophe and the tritronquée solution to the Painlevé I equation *J. Nonlinear Sci.* **19** 57–94
- [DT00] Delabaere E and Trinh D T 2000 Spectral analysis of the complex cubic oscillator *J. Phys. A: Math. Gen.* **33** 8771–96
- [Elf34] Elfving G 1934 Über eine Klasse von Riemannschen Flächen und ihre Uniformisierung *Acta Soc. Sci. Fenn. N.s.* **2** 3 1–60
- [Fed93] Fedoryuk M 1993 *Asymptotic Analysis* (Berlin: Springer)
- [FMZ92] Fokas A, Muğan U and Zhou X 1992 On the solvability of Painlevé I, III and V *Inverse Problems* **8** 757–85
- [GLS00] Gromak V I, Laine I and Shimomura S 2000 *Painlevé Differential Equations in the Complex Plane* (de Gruyter & Co)
- [IN86] Its A and Novokshenov V 1986 *The Isomonodromic Deformation Method in the Theory of Painlevé equations* (Berlin: Springer)
- [Inc56] Ince E L 1956 *Ordinary Differential Equations* (New York: Dover)
- [JK88] Joshi N and Kruskal M 1988 An asymptotic approach to the connection problem for the first and the second Painlevé equations *Phys. Lett. A* **130** 129–37
- [JK01] Joshi N and Kitaev A V 2001 On Boutroux's tritronquée solutions of the first Painlevé equation *Stud. Appl. Math.* **107** 253
- [Kap04] Kapaev A A 2004 Quasi-linear Stokes phenomenon for the Painlevé first equation *J. Phys. A: Math. Gen.* **37** 11149

- [Kit94] Kitaev A V 1994 Elliptic asymptotics of the first and the second Painlevé transcendents *Russ. Math. Surv.* **49** 81–150
- [KK93] Kapaev A A and Kitaev A V 1993 Connection formulae for the first Painlevé transcendent in the complex domain *Lett. Math. Phys.* **27** 243
- [KT05] Kawai T and Takei Y 2005 *Algebraic Analysis of Singular Perturbation Theory* (Providence, RI: American Mathematical Society)
- [Nev70] Nevanlinna R 1970 *Analytic Functions* (Berlin: Springer)
- [Nov09] Novokshenov V 2009 Padé approximants for Painlevé I and II transcendents *TMF* **159** 515–26
- [Olv74] Olver F 1974 *Asymptotics and Special Functions* (New York: Academic)
- [Sim70] Simon B 1970 Coupling constant analyticity for the anharmonic oscillator. (With appendix) *Ann. Phys.* **58** 76–136
- [Str84] Strebel K 1984 *Quadratic Differentials* (Berlin: Springer)